J. Fluid Mech. (1968), *vol.* 32, *part* 3, *pp.* 437–447
Printed in Great Britain

Vorticity expulsion by turbulence: astrophysical implications of an Alka-Seltzer experiment

By D. O. GOUGH
Department of Astronomy, Columbia University, New York
& D. LYNDEN-BELL
Royal Greenwich Observatory, Herstmonceux Castle, Hailsham, Sussex

(Received 10 May 1967 and in revised form 19 December 1967)

Arguments are given which suggest that the mean flow of a turbulent fluid is irrotational far from the boundaries. Mathematical difficulties precluded a detailed theory so a simple experiment was performed. This experiment qualitatively confirmed the suggestion. Other well-known experiments on the mean flow of turbulent fluid may also be interpreted in terms of vorticity expulsion, an idea closely related to Scorer's hypothesis on turbulent mixing.

Theoretical papers on stellar rotation usually assume that convective regions rotate uniformly because of a high turbulent viscosity. However, if vorticity expulsion occurs, convective stellar cores will not rotate. The concept of a turbulent viscosity is criticized.

1. Arguments for vorticity expulsion

(*a*) At least for barotropic and incompressible fluids the lines of vorticity move with the fluid. Over and above this there will be a slow diffusion of vortex lines through the fluid due to molecular viscosity.

Consider an arbitrary set of lines that pass through the fluid initially in some direction and let them move with the fluid throughout its turbulent history. After some time it is difficult to see how any general sense of direction will be preserved.

Although locally there is a flux of lines across a simple surface one might expect that the mean flux of lines across a large surface spanning many turbulent elements should vanish. This would lead us to the conclusion that the mean velocity of the fluid was irrotational.

It is clear that the above argument is woolly; that it must be wrong in the case of an isolated fluid where angular momentum is preserved, and that it makes no allowance for the fact that the lines which we are really interested in are very special ones, for the flow—the vortex lines.

(*b*) Our interest in this subject first arose from considerations of convection in differentially rotating stars. To simplify our argument we will here consider an unperturbed fluid of constant entropy density. The force due to rotation is of magnitude mh^2R^{-3} directed outward from the axis. m is the mass of the element considered, h its angular momentum per unit mass and R its distance from the axis. This force is derivable from a potential energy $\frac{1}{2}mh^2R^{-2}$ where h is the fixed value for the fluid element considered.

Let two elements each of mass m have angular momenta h_1, h_2 per unit mass and let them be situated at R_1 and R_2 respectively. Then an energy

$$\Delta E = m\frac{h_1^2-h_2^2}{2}\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right) \tag{1}$$

will be released in a fluid motion which exchanges them while conserving their angular momenta. [The constant entropy density ensures that there is no change of internal energy.] (Rayleigh 1916, cf. 1880; Howard & Gupta 1962).

ΔE is positive if h^2 decreases outwards, which is Rayleigh's well known criterion for instability. The motions that result from the instability tend to rearrange things so that h^2 no longer decreases outwards. However, they cease once h^2 is constant throughout the fluid. Since for toroidal motions the vorticity is given by

$$\boldsymbol{\omega} = \operatorname{curl}\mathbf{u} = \left(0, 0, \frac{1}{R}\frac{\partial h}{\partial R}\right), \tag{2}$$

the constancy of h is equivalent to the complete expulsion of vorticity.

Now consider the opposite case when Rayleigh's criterion for instability is not satisfied but when motions which exchange elements of fluid are forced by an imposed turbulence. The constant interchange of elements of fluid with different values of h must lead to dh^2/dR decreasing towards zero. Thus again we are led to the belief that the turbulence leads towards constant h for the mean flow.

(*c*) To clarify the mechanism by which vorticity is expelled, it is useful to use Batchelor's (1950) analogy between vorticity and magnetic field. Weiss (1966) has shown that magnetic field is expelled from inexorable eddies in a conducting fluid. If the magnetic Reynold's number is high, the field inside the eddy is stretched and amplified; it reconnects and eventually diffuses away, leaving only the field outside. It is important to understand the steady state in which the convection and the diffusion of the vector field balance from a physical viewpoint. Why does not the magnetic field diffuse back into the eddy?

Weiss's conclusion that the magnetic field is expelled is strikingly confirmed by Parker's (1966) exact solution for the rigidly rotating eddy. We suggest that vorticity experiences a similar process to the magnetic field. If the turbulence is sufficiently energetic an eddy will concentrate the vortex lines transverse to it, at its edges and twist them up inside. At high Reynolds numbers the viscosity will cause the expelled vortex lines to become disconnected from those inside the eddy which have now become part of the turbulence. The expelled lines may execute a random walk along the eddy interstices until they reach the walls or edges of the turbulent region.

It should be emphasized that it is absolute vorticity (relative to inertial axes) which suffers this process. To stress this Dr Ostriker has likened the eddies to liquid gyrostats.

Although the analogy between vorticity and magnetic field is not complete, it is likely that if the forces generating the turbulence are sufficiently large, the kinematical calculations by Weiss will have some relevance; and quantitative estimates pertinent to magnetic field expulsion will have some bearing on the expulsion of vorticity. Of particular interest is the time τ_e taken for vorticity to be expelled and disconnected from that remaining in an eddy. We estimate this in a manner similar to Weiss (1966). If the eddy turnover time is τ_0 then that part of the vorticity that is transverse to the main eddy gets a complete coil in this time. Neglecting diffusion effects the same strand of transverse vorticity will now cross the eddy five times instead of once. Calling the transverse vorticity B within the strand we have

$$B = B_0(1+4t/\tau_0), \tag{3}$$

which in differential form yields

$$\frac{dB}{dt} = \frac{4}{\tau_0}\frac{B}{1+4t/\tau_0}. \tag{4}$$

By reason of the same coiling, the length scale over which the transverse vorticity changes direction is reduced from L the diameter of an eddy to

$$d = L(1+4t/\tau_0)^{-1}. \tag{5}$$

Now the diffusion time due to finite viscosity is proportional to $d^2\nu^{-1}$ where ν is the kinematic viscosity. The constant of proportionality may be evaluated in the simplest cases to be about π^{-2}. Since opposing strands annihilate, due to diffusion, the rate of change due to diffusion is

$$\left(\frac{dB}{dt}\right)_{\text{diffusion}} = -\frac{\pi^2\nu}{d^2}B = \frac{-\pi^2\nu}{L^2}(1+4t/\tau_0)^2 B. \tag{6}$$

Hence including both convection and diffusion we obtain

$$\frac{\tau_0}{4}\frac{dB}{dt} = \frac{B}{1+4t/\tau_0} - \frac{\pi^2\nu\tau_0}{4L^2}\left(1+\frac{4t}{\tau_0}\right)^2 B. \tag{7}$$

This integrates to give

$$B = B_0(1+4t/\tau_0)\exp\left[\frac{1-(1+4t/\tau_0)^3}{12\pi^{-2}R_e}\right], \tag{8}$$

where the Reynolds number R_e is $L^2\nu^{-1}\tau_0^{-1}$.

Equations (7), (8) show that the transverse vorticity increases to a strength of about $B_0(R_e\pi^{-2})^{\frac{1}{3}}$ when the reconnexion occurs and rapidly annihilates the field inside. Thus we may expect the expelled vorticity to have a total flux L^2B_0 and a magnitude $B_0(R_e\pi^{-2})^{\frac{1}{3}}$, and that the expulsion occurs after a time $\frac{1}{2}(R_e\pi^{-2})^{\frac{1}{3}}\tau_0$. We shall use these order of magnitude estimates in a rough comparison with experiment. Note that they assume a persistence of the eddy for times greater than τ_0, a condition which will not be properly satisfied in practice.

(*d*) If two elements of different angular momentum per unit mass are thoroughly mixed by the turbulence then the resulting fluid element will have the total mass and total angular momentum of the original elements. Thus its angular momentum per unit mass will be intermediate. After many such mixings one might expect a uniformity of angular momentum per unit mass h within the fluid. Since the vorticity is proportional to the gradient of h the turbulent part of the fluid will be irrotational. This type of argument was used by Scorer (1965, 1966).

Rigorously it depends on the fact that the probability of an element being mixed with another element of different h is independent of the angular momentum of the first element. If biased elements suffered mixing the argument could break down.

Bretherton & Turner (1968) show that there is such a systematic bias in the motions of gas molecules which are assumed to be isotropic about a mean at each point. Those molecules in the inner parts, which have their peculiar motions oriented to give them a higher than average angular momentum, preferentially move out and interact with molecules further out. Bretherton & Turner show that under certain other restrictions this effect leads to solid body rotation as the equilibrium state of a gas with isotropic velocity dispersion.

In connexion with anisotropy it is interesting to point out that in encounterless stellar dynamics where there are differentially rotating equilibria the velocity dispersions in the radial and tangential relations obey the relation (Lindblad 1959):

$$\frac{\langle V R^2\rangle}{\langle (V_\phi - \langle V_\phi\rangle)^2\rangle} = \frac{2\Omega}{\omega} = \frac{2\langle V_\phi\rangle / R}{\frac{1}{R}\frac{d}{dR}(R\langle V_\phi\rangle)},$$

so that here too vanishing of the anisotropy leads to solid body rotation.

Do turbulent motions merely mix or do they have the same built-in bias as the molecular motions that cause viscosity? Are turbulent motions isotropic about the mean even in the presence of Coriolis force?

2. The experiment

A 1 l. beaker containing 350 ml. carbon tetrachloride on which was floating a layer of about 150 ml. water, the working fluid, was floated in the middle of a 4 l. beaker of water, and the whole was set on a rotating table (figure 1). The inner beaker was thus supported by a frictionless bearing when the two beakers were co-rotating. The carbon tetrachloride prevented any tornado formed in the water from transmitting a high torque to the base of the inner beaker. A thin layer of water was used so as to minimize large-scale meridional circulation. A scale on the rim of the outer beaker and a pointer on the inner beaker enabled the relative orientations of the two beakers to be measured.

When the system had achieved a state of rigid-body rotation, turbulence was generated by dropping from a rotating frame several tablets of Alka-Seltzer† into the working layer of water. The principle of the experiment is clear: if turbulence does expel vorticity, and with it angular momentum, to the periphery, the viscous stress at the wall will cause the rotation rate of the inner beaker to increase. Such an increase was consistently observed.

Many criticisms may be levelled against the experiment in this crude form. Below are listed some of the most important, and the refinements that were made to test or remove factors which might have made the results spurious. None of the alterations made any qualitative difference to the result.

† Alka-Seltzer is an analgesic in tablet form which derives its popularity from the effervescent reaction between citric and acetylsalicylic acids and sodium bicarbonate when dissolved in water.

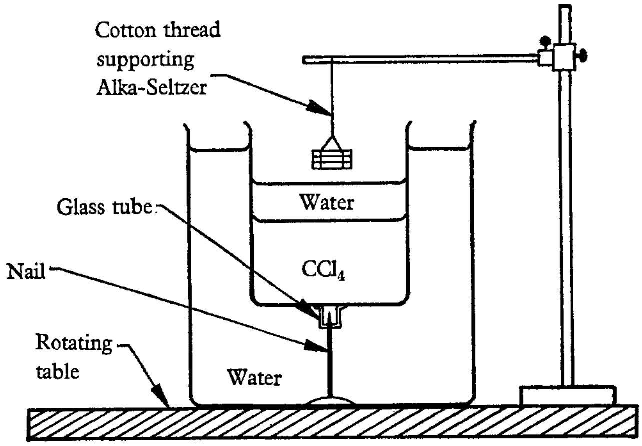

FIGURE 1. Diagram of the apparatus in its original form.

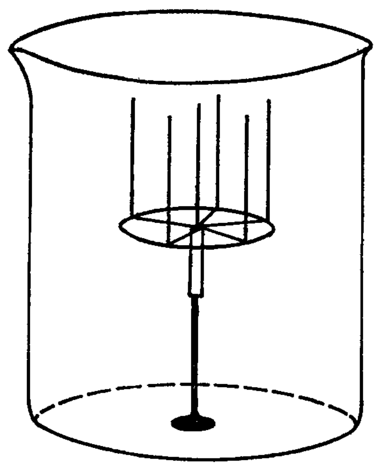

FIGURE 2. The inner beaker, with the wire frame which supported the Alka-Seltzer.

(i) A small piece of glass tubing was attached to the underside of the inner beaker and fitted closely around a nail projecting upwards from the base of the outer beaker (figure 1). The two beakers were thus constrained to rotate about a common axis, and any wander which might give rise to relative rotation was eliminated.

(ii) Originally the Alka-Seltzer was suspended by a thread from a rotating stand and dropped into the water by burning the thread. This had two obvious disadvantages. The tablets, which initially floated on the carbon tetrachloride, became saturated with this inert liquid and ceased to react. They also tended to congregate by the wall of the beaker and cause large-scale currents. The resultant non-uniform distribution of bubbles changed the moment of inertia of the water, but this, of course, was temporary. These troubles were overcome by making a wire frame which pivoted on a second nail inside the inner beaker (figure 2).

Holes were drilled in the Alka-Seltzer tablets which were then threaded onto the wires and supported just above the water's surface by a single thread. When the thread was burnt, all the tablets fell vertically together and came to rest within the water layer.

(iii) It was hoped to observe a decrease in the rotation rate of the frame, but the friction at the pivot was too great.

(iv) The weight of the Alka-Seltzer on the frame was borne by the inner beaker. During the reaction there was slight mass loss due to escaping carbon dioxide; the inner beaker became more buoyant and rose, presumably causing circulation in the surrounding water. But when the Alka-Seltzer was originally dropped from the stand, there was a mass increase.

(v) The inner beaker was not perfectly axisymmetric and did not float vertically. Small lead weights were placed on its rim to correct for this with no detectable difference in the results.

(vi) It was not always possible to achieve a state of rigid-body rotation at the outset, since a slight temperature difference between the contents of the two beakers was produced from the endothermic reaction of the previous run. This gave rise to a weak circulation. Nevertheless, the initial drift was always much smaller than the subsequent relative rotation. A Perspex lid to reduce cooling from evaporation had little effect.

(vii) Over the course of an hour the speed of the turntable was found to fluctuate by less than 1 %.

(viii) An important limitation of the experiment was that it was not possible to achieve a steady state, nor to measure the mean velocity distribution of the turbulent layer. It may be practical to design a steady experiment generating bubbles by electrolysis between two horizontal platinum grids, but there remains the worry that the bubbles themselves may influence the dynamics. It would also be desirable to measure how the angular momentum transferred depends on viscosity and the properties of the turbulence.

3. Experimental results

Whatever the modification made the result was always qualitatively the same: as soon as the violent effervescence had ceased there was a marked increase in the rotation rate of the inner beaker. Five experiments were run with the apparatus in its final form. The results are shown in table 1. While the Alka-Seltzer was effervescing, the inner beaker oscillated slightly about its mean position, sometimes showing signs of increasing its rotation rate. But it was not until the water was relatively free from bubbles that a marked acceleration was observed. Later, the inner beaker gradually slowed down as angular momentum was transferred to the carbon tetrachloride and the water outside. Tracer particles of potassium permanganate showed that the water layer was still turbulent 10 min after the effervescence had ceased. The reason is that a layer of powdered Alka-Seltzer was deposited on the carbon tetrachloride surface, and continued to issue small bubbles long after the tablets had disappeared.

In figure 3 the velocity excess of the inner beaker in experiment 5 is plotted against time. Some of the scatter is real, but some is due to the difficulty experienced in measuring the relative orientation of the moving beakers, and probably explains the high incidence of points at 10 and 15 degrees per period. But the trend is definite. In this run the inner beaker made nearly two and a half revolutions more than the turntable.

Experiment no.	Initial drift (deg. sec^{-1})	Max. excess rotation (deg. sec^{-1})	Approx. time when achieved (sec)	No. of Alka-Seltzer used
1	None	0·75	300	12
2	None	0·74	200	6
3	0·3	0·96	250	6
4	None	1·25	300	12
5	0·1	1·15	300	18

TABLE 1. Analysis of results. 165 ml. water was used in the first four experiments and 200 ml. in the fifth. Slightly warmer water was used for the working fluid in experiment 4 to halt the initial drift. The rotation rate of the turntable was 23·2 deg. sec^{-1}. The fourth column shows the time taken for the inner beaker to reach its maximum velocity

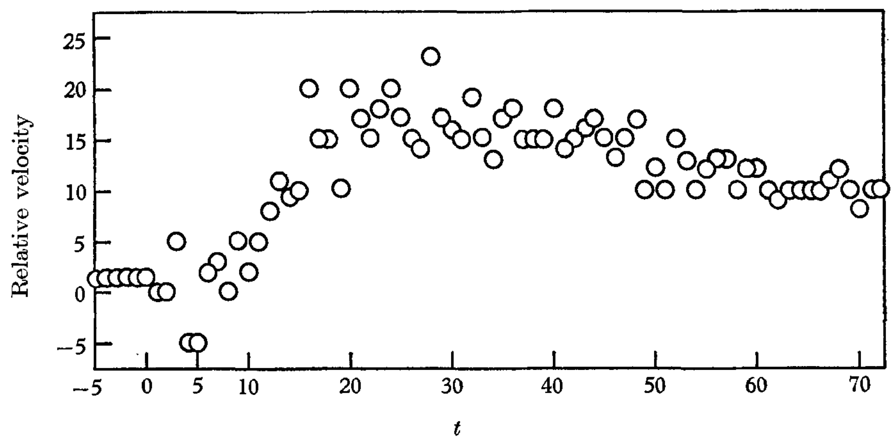

FIGURE 3. Variation of the relative velocity of the inner beaker with time. Time is measured in periods of the inner beaker and velocity in degrees per period. The Alka-Seltzer was dropped at $t = 0$.

It was not easy to obtain good estimates of typical length and time scales of the turbulence. Watching tracer particles was probably as good a method as any; order-of-magnitude estimates of $L = 1$ cm and $\tau_0 = 1$ sec were made for the largest eddies. These are to be compared with the diameter of the beaker, $D = 10{\cdot}2$ cm, and the period of the turntable, $T = 15{\cdot}5$ sec. Since the bulk viscosity, μ of water, is about 0·01 poise, the Reynolds number R_e was approximately 100.

During the acceleration, the turbulent water by the beaker wall was rotating noticeably faster than the beaker itself. In some runs, but not all, a slight large-scale meridional circulation could be detected when the layer became quiescent.

When the turbulence had ceased, there remained circular eddy-like patterns in the powder that had been deposited on the surface of the carbon tetrachloride. There were about twenty, with diameters of about 1 cm, close to the edge of the beaker. In the middle the patterns were larger, irregular, and less distinct.

Comparison with the theoretical estimate

We shall use the results of experiment 5. If the description proposed above of the vorticity expulsion mechanism is correct, there are three distinct processes that occur. First, the vorticity must be expelled from the eddies. This occurs in a time of order $(\pi^{-2}R_e)^{\frac{1}{3}}\tau_0 = 2\,\text{sec}$ and is short compared to the time (300 sec) during which the acceleration took place. On expulsion, the filaments execute a random walk between the eddies and accumulate near the walls of the beaker. This time is of order $(D/2L)^2\tau_0 = 25\,\text{sec}$ which is again short. Finally, the viscous torque on the wall transfers angular momentum to the beaker.

According to our estimates, we would expect the expelled vortex filaments to have vorticity

$$\omega = (R_e\pi^{-2})^{\frac{1}{3}}\omega_0$$

and diameters

$$\lambda = (R_e\pi^{-2})^{-\frac{1}{6}}L,$$

where ω_0 is the initial vorticity of the layer $(4\pi/T)$. If there are $\pi D/L$ vortex filaments at the wall at any time, one for each eddy, the mean torque on the wall is approximately

$$\begin{aligned} G &= \pi D(\lambda/L)\,\tfrac{1}{2}D\Delta\mu\omega \\ &= 2V\mu\omega_0(\pi^{-2}R_e)^{\frac{1}{6}} \\ &\simeq 5{\cdot}0\,\text{cm dyne}, \end{aligned}$$

where μ is the viscosity, Δ the depth, and V the volume of the turbulent water. The acceleration of the beaker can now be estimated as

$$G/I \simeq 7{\cdot}0\times 10^{-4}\,\text{sec}^{-2},$$

where $I = 7000\,\text{g cm}^2$ is the moment of inertia of the beaker. Of course, this will considerably overestimate the acceleration because the transfer of angular momentum to the carbon tetrachloride and the water in which the inner beaker was floating has been neglected.

The maximum acceleration in experiment 5 can be estimated from figure 3 to be about $10^{-4}\,\text{sec}^{-2}$, which differs from the theoretical estimate by an order of magnitude in the expected sense.

Other experiments

Emboldened with this apparent success we looked for other experiments which had received incomplete but plausible conventional explanations, but which could also throw light on the mean vorticity of a turbulent fluid.

Taylor's (1935) experiments with turbulence between co-axial cylinders showed that the mean vorticity is completely expelled to the walls where thin boundary layers formed.

Further demonstration of vorticity expulsion is provided by flow through a pipe; when the flow becomes turbulent, the mean velocity is uniform except near the wall. Expulsion of angular momentum has been observed by Fultz *et al.* (1959) during their dishpan experiments. In the Hadley (axi-symmetric) regime, the angular momentum per unit mass exceeded $r_0^2\Omega$ (where r_0 is the radius of the pan and Ω its angular velocity) at distances greater than $0{\cdot}8r_0$ from the centre. It was suggested that this may have been caused by some eddy production mechanism. Space and time fluctuations were present, although the intensity was much lower than in the Rossby regime, 'especially in the outside half of the pan' where the stabilizing influence of the mean flow was then greatest.

4. Scorer's hypothesis

In an attempt to explain the formation of hurricanes, Scorer (1965, 1966) considered the effect of turbulence on a rotating mass of air. He proposed that angular momentum is transported radially by large eddies and then mixed into the surroundings by smaller scale turbulence. In this way a uniform distribution of angular momentum is set up in a well-stirred fluid and a strong vortex is concentrated at the centre of activity. For atmospheric motions on a scale relevant to hurricanes, the vortex should form in about 2 days.

Scorer implicitly assumes that the turbulence is dominated by radial velocities directed towards some origin. This would indeed produce an influx of angular momentum (e.g. Biermann 1951), but it is difficult to see why such an organized stirring should spontaneously arise. A more detailed discussion is given by Bretherton & Turner (1968) who also conducted an experiment to test Scorer's principle. Turbulence was generated in a rotating layer of paraffin oil by means of an oscillating rigid grid. An attempt to detect a mean drift by following neutrally buoyant marker particles failed, and it was concluded that Scorer's mechanism, if operative, is less than 5 % efficient.

It is necessary to reconcile this result with the conclusions of the present investigation. The presence of the grid constrained the mean flow of the paraffin to a high degree; the decay time of large-scale swirl was found to be a 'few seconds' when the grid was oscillating but was 'several minutes' when the grid was stationary. Any vorticity diffusion must be effective in a few seconds, therefore, if it is to be detected. But the quoted typical length and velocity scales, 5 and 10 $\mathrm{cm\,sec^{-1}}$, and a kinematic viscosity of about 0·02 stokes yield an expulsion time of several seconds.

We conclude further that the mechanism we envisage is not relevant to hurricane formation. The length and velocity scales quoted by Scorer give an expulsion time of 10^8 sec, several years.†

A referee pointed out that if an eddy viscosity was used in our estimates rather than a molecular viscosity then expulsion times of the order of a few days would be obtained and these are consistent with hurricane formation times. The mechanism by which vortex lines are reconnected by an eddy viscosity is not simple, and we think that straight application of our formulae to this case is even more doubtful than our argument. Presumably an eddy viscosity is a way of describing complete cycles of eddy winding, diffusion and random walk of detached vortex lines all going on at smaller scales than the large eddies. In any case our ideas would suggest a random walk of expelled vortex lines leading to their dispersal rather than to the formation of a giant vortex.

† The theory of § 1 gives 10^6 sec. A three-dimensional generalisation is only of much importance here so we have included an amended number.

5. Astrophysical applications

More conventional ideas on hurricanes do provide an interesting example of a limitation on the generality of vorticity expulsion. When water is boiled in a rotating bucket a strong meridional circulation occurs giving a rapidly rotating rising column and a slowly rotating fall. This flow arises from temperature difference both along the bottom and on the sides. Circulation inside stars could well be driven by a similar process, originating in a convective zone.

However, if such strong circulations are not set up then we expect convective zones to show vorticity expulsion. Since the axis is included in a convective core those cores should not rotate.

Applications of these ideas to the Hayashi phase might provide a means by which stars shed high angular momentum material (Hayashi 1961).

It is important to realize that the oft used concept of a turbulent viscosity has severe limitations. Consider the case of a rotating cylinder experiment in which the motions are very slow and laminar. The complete solution is given by use of the molecular viscosity. If now exactly the same experiment is set up in a uniformly rotating laboratory the velocities with respect to that laboratory will be the same as those in the first experiment.

However, perform the same two experiments with the inner cylinder rotating so fast that in both experiments the flow is highly turbulent and both will show a region in which $\langle V(R)\rangle R$ is constant where the $\langle V(R)\rangle$ is the average toroidal velocity *measured with respect to inertial axes in both cases.* This is a very different result from that predicted by analogy with the laminar flow experiments.

The ill-explained differential rotation of the sun and the general circulation of the earth's atmosphere need some systematic vorticity expulsion to explain how the equatorial regions are maintained with both the greatest angular momentum per unit mass and the greatest angular velocity of any part of the body.

It is hoped that this work will stimulate more and better experiments on the interaction of turbulence and rotation.

The authors would like to express their gratitude to many members of the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, who discussed this problem, especially to T. B. Benjamin who suggested the experiment and to F. P. Bretherton and N. O. Weiss. D.O.G. is grateful to the Science Research Council for award of a research studentship.

REFERENCES

Batchelor, G. K. 1950 *Proc. Roy. Soc.* A, **201**, 405.

Biermann, L. 1951 *Zs. f. Ap.* **28**, 304.

Bretherton, F. P. & Turner, J. S. 1968 *J. Fluid Mech.* **32**, 449.

Fultz, D., Long, R. L., Owens, G. V., Bohan, W., Kaylor, R. & Weil, J. 1959 *Met. Mon.* **4**, no. 21, 26.

Hayashi, C. 1961 *Pub. Astr. Soc. Japan,* **13**, 450.

Howard, L. N. & Gupta, A. S. 1962 *J. Fluid Mech.* **14**, 463.

Lindblad, B. 1959 *Hand. Phys.* **53**, 31.

PARKER, R. L. 1966 *Proc. Roy. Soc.* A, **291**, 60.
RAYLEIGH, LORD 1880 *Proc. Lond. Math. Soc.* **14**, 170 (*Sci. Papers* **2**, 200).
RAYLEIGH, LORD 1916 *Proc. Roy. Soc.* A, **93**, 148 (*Sci. Papers* **6**, 447).
SCORER, R. S. 1965 Unpublished Report (Imperial College).
SCORER, R. S. 1966 *Sci. J.* **2**, 46.
TAYLOR, G. I. 1935 *Proc. Roy. Soc.* A, **151**, 494 (*Sci. Papers* **2**, 336).
WEISS, N. O. 1966 *Proc. Roy. Soc.* A, **293**, 310.